Advanced ODE-Lecture 14
Lyapunov Theory for Time-varying Systems

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- **Equivalent Definitions for Lyapunov Stability**
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Motivation

- Tracking problem in control: Suppose that $x_0(t)$ is a solution of $x' = g(x)$ as a reference trajectory. Let $e = x - x_0(t)$. Then,

$$e' = x' - x_0'(t) = g(x) - g(x_0(t)) = g(e + x_0(t)) - g(x_0(t)) := f(t, e).$$

If the equilibrium $e = 0$ of the error equation $e' = f(t, e)$ is AS, then we say that x tracks a reference trajectory $x_0(t)$ (for example, $x_0(t)$ is a limit cycle). The origin $e = 0$ of the time-varying system $e' = f(t, e)$ corresponds to the reference solution of the time-invariant system $x' = g(x)$! Although the reference system is autonomous, we have to solve a time-varying system for a tracking problem.

- There are many problems， like nonholonomic systems, that are autonomous. They need a time-varying feedback to control such systems, so their closed-loop systems are time-varying systems. Nonholonomic problems have many applications in control, like rigid robot models, underwater ship models, etc. This is a hot topic of research in control and applications.

- In Math itself, time-varying systems are a natural extension of autonomous systems. However, the three fundamental properties of autonomous systems are no longer true for time-varying systems. So the study of time-varying systems is not trivial, they will be much more complicated and tougher than we expect.

Equivalent Definitions for Lyapunov Stability

Consider the time-varying system

$$\dot{x} = f(t, x), \qquad (14.1)$$

where f is continuous and locally Lip. in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain containing the origin, and $f(t, 0) \equiv 0$, $\forall\ t \geq 0$.

1) Comparison Function: Class K and Class KL

Definition 14.1 A continuous function $\alpha: [0,a) \to [0,\infty)$ is of class K if it is strictly increasing and $\alpha(0) = 0$, denoted by $\alpha \in K$. It is class K_∞ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, $\alpha \in K_\infty$.

Definition 14.2 A continuous function $\beta: [0,a) \times [0,\infty) \to [0,\infty)$ is of class KL if, for each fixed s, $\beta(r,s)$ is class K w.r.t. r and, for each fixed r, $\beta(r,s)$ is decreasing w.r.t. s and $\beta(r,s) \to 0$ as $s \to \infty$.

Example 14.1

- $\alpha(r) = \tan^{-1} r \nearrow$ since $\alpha'(r) = \dfrac{1}{1+r^2} > 0$. It is of class K, but not of class K_∞ since $\lim_{r\to\infty} \alpha(r) = \dfrac{\pi}{2} < \infty$.
- $\alpha(r) = r^c \nearrow$ for any $c > 0$ since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$. So, it is of class K_∞.
- $\beta(r,s) = \dfrac{r}{ksr+1} \nearrow$ in r for any $k > 0$ since $\dfrac{\partial\beta}{\partial r} = \dfrac{1}{(ksr+1)^2} > 0$ and $\searrow$ in s since

$$\frac{\partial\beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0.$$

  Moreover, $\beta(r,s) \to 0$ as $s \to \infty$. Hence, it is of class KL.
- $\beta(r,s) = r^c e^{-s}$, for $c > 0$, is of class KL.

Lemma 14.1 Let $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ be of class K functions on $[0,a)$, $\alpha_3(\cdot)$ and $\alpha_4(\cdot)$ be of class K_∞ functions, and $\beta(r,s)$ be of class KL function. Then, $\alpha_j^{-1}(\cdot)$ is defined on $[0,\alpha_1(a))$ and is of class K; $\alpha_3^{-1}(\cdot)$ is defined on $[0,\infty)$ and is of class K_∞; $\alpha_1 \circ \alpha_2$ is of class K; $\alpha_3 \circ \alpha_4$ is of class K_∞; $\sigma(r,s) = \alpha_1(\beta(\alpha_2(r),s))$ is of class KL.

Remark 14.1 Class K and KL functions are important tools in analysis of nonlinear systems.

2) Equivalent Definitions of Uniform Stability

Lemma 14.2 The origin $x=0$ of (14.1) is

- uniformly stable (US in short) $\Leftrightarrow$ $\exists$ $\alpha(\cdot)\in K$ and $c>0$, independent of $t_0 \ge 0$, such that

$$\| x(t;t_0,x_0)\| \le \alpha(\|x_0\|),\ \ \forall\ t\ge t_0\ge 0,\ \ \forall\ \|x_0\|<c; \tag{14.2}$$

- uniformly asymptotically stable (UAS in short) $\Leftrightarrow$ $\exists$ $\beta(r,s)\in KL$ and $c>0$, independent of $t_0\ge 0$, such that

$$\| x(t;t_0,x_0)\| \le \beta(\|x_0\|,\ t-t_0),\ \ \forall\ t\ge t_0\ge 0,\ \ \forall\ \|x_0\|<c; \tag{14.3}$$

- uniformly globally asymptotically stable (UGAS in short) $\Leftrightarrow$ (14.3) is satisfied for any $x_0\in R^n$.

- exponentially stable (ES) if (14.3) is satisfied with $\beta(r, s) = kre^{-\gamma s}$.
- globally exponentially stable (GES) if (14.3) is satisfied with $\beta(r, s) = kre^{-\gamma s}$ for any $x_0 \in R^n$.

Remark 14.2 For time-varying systems, GAS and UGAS are different. For example,

$$x' = -\frac{x}{1+t}$$

has a solution $x(t) = x_0 \frac{1+t_0}{1+t}$. It is GAS. However, it is not UGAS by contradiction.

If there exists $\beta \in KL$ s.t. for all $t \ge t_0 \ge 0$, $|x(t)| = |x_0 \frac{1+t_0}{1+t}| \le \beta(|x_0|, t-t_0)$ could be satisfied, we would have

$$\frac{1}{2} = \frac{1+t_0}{2+2t_0} \le \beta(1, 1+t_0)$$

by taking $x_0 = 1$ and $t = 2t_0 + 1 > t_0 \ge 0$. Since $\beta(1, 1+t_0) \to 0$ as $t_0 \to \infty$, this contradiction shows that GAS and UGAS are different.

3) Some Important Auxiliary Results

Lemma 14.3 Consider a scalar equation

$$y' \le -\alpha(y), \quad y(t_0) = y_0$$

where $\alpha(\cdot) \in K$ is a locally Lip., defined on $[0, a)$. For all $0 \le y_0 \le a$, this equation has a unique solution defined for all $t \ge t_0$. Moreover,

$$y(t) \le \sigma(y_0, t - t_0),$$

where $\sigma(r, s) \in KL$ defined on $[0, a) \times [0, \infty)$.

Remark 14.3 This comparison lemma is very useful for analysis of Lyapunov stability. The proof itself is simply application of the comparison principle.

Lemma 14.4 For any $V(x) > 0$ (positive definite), where $x \in D \subset R^n$, and $B_r \subset D$ where $r > 0$, then, $\exists$ $\alpha_1(\cdot), \alpha_2(\cdot) \in K$, defined on $[0, r)$, such that

$$\alpha_1(\| x \|) \le V(x) \le \alpha_2(\| x \|) \qquad (14.4)$$

for all $x \in B_r$. Moreover, if $D = R^n$ and $V(x)$ is radially unbounded, then $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ can be chosen to be of class K_∞ and (14.4) holds for all $x \in R^n$.

Remark 14.4 All the proofs of these Lemmas can be found in "Nonlinear Systems" 3rd ed. by H. Khalil, Prentice Hall, Upper Saddle River, NJ, 2002. Here all are omitted.

4) Lyapunov Theorem for Time-Varying Systems

Theorem 14.1 Let $V:[0,\infty)\times D \to R$ be of C^1 such that

$$W_1(x) \le V(t,x) \le W_2(x)\,; \tag{14.5}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le 0\,, \tag{14.6}$$

for all $t \ge 0$, and all $x \in D$, where $W_j(x)$ ($j = 1, 2$) are positive definite. Then, $x = 0$ of (14.1) is US. If $D = R^n$, and $W_1(x)$ is radially unbounded, then the origin is uniformly globally stable.

Proof. Since the derivative of $V(t, x)$ along trajectories of (14.1) is given by

$$V'(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le 0 ,$$

we choose $r > 0$ and $\rho > 0$ such that $B_r \subset D$ and $\rho < \min_{\|x\| = r} W_1(x)$. Then,

$$\{x \in B_r \mid W_1(x) \le \rho\} \subset B_r .$$

Define a time-dependent set $\Omega_{t,\rho}$ by

$$\Omega_{t,\rho} = \{x \in B_r \mid V(t, x) \le \rho\} .$$

Since $W_2(x) \le \rho \;\Rightarrow\; V(t,x) \le \rho$, we have

$$\{x \in B_r \mid W_2(x) \le \rho\} \subset \Omega_{t,\rho} .$$

On the other hand, $V(t,x) \le \rho \;\Rightarrow\; W_1(x) \le \rho$ yields $\Omega_{t,\rho} \subset \{x \in B_r \mid W_1(x) \le \rho\}$.

Thus,

$$\{x \in B_r \mid W_2(x) \le \rho\} \subset \Omega_{t,\rho} \subset \{x \in B_r \mid W_1(x) \le \rho\} \subset B_r \quad \text{for all} \quad t \ge 0\,.$$

These five nested sets are sketched in Fig. 14.1.

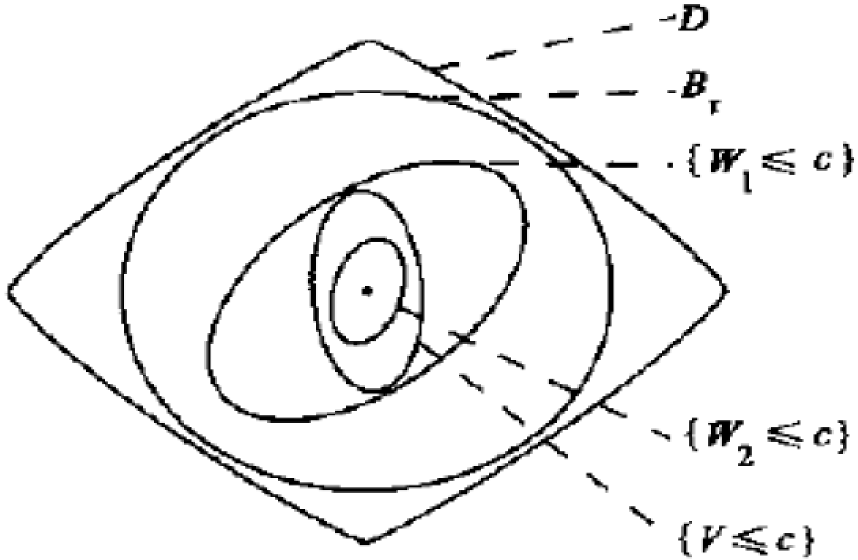

Fig. 14.1

For any $t_0 \ge 0$, and any $x_0 \in \Omega_{t_0,\rho}$, the solution $x(t;t_0,x_0)$ stays in $\Omega_{t,\rho}$ for all $t \ge t_0$ because $V'(t,x(t;t_0,x_0)) \le 0 \quad \Rightarrow \quad V(t,x(t;t_0,x_0)) \le V(t_0,x_0) \le W_2(x_0) \le \rho$ for all $t \ge t_0$. Therefore, $x(t;t_0,x_0) \in B_r$ and $x(t;t_0,x_0)$ is defined for all $t \ge t_0$ by Extensibility theorem.

By Lemma 14.4, there exist α_1 and $\alpha_2 \in K$, defined on $[0, r]$, such that

$$W_1(x) \geq \alpha_1(\| x \|), \quad W_2(x) \leq \alpha_2(\| x \|).$$

Then, we have

$$\alpha_1(\| x(t; t_0, x_0) \|) \leq V(t, x(t; t_0, x_0)) \leq V(t_0, x_0) \leq W_2(x_0) \leq \alpha_2(\| x_0 \|).$$

From which we conclude that

$$\| x(t; t_0, x_0) \| \leq \alpha_1^{-1}(\alpha_2(\| x_0 \|)) = \alpha(\| x_0 \|),$$

where $\alpha \in K$ by Lemma 14.1. Therefore, the origin is US by Lemma 14.2.

If $D = R^n$, $\alpha_1, \alpha_2 \in K_\infty$. Hence, α_1 and α_2 are independent of t_0. Since $W_1(x)$ is radially unbounded, we can choose $\rho > 0$ such that $\| x_0 \| \leq \alpha_2^{-1}(\rho)$. Then, $x_0 \in \{x \in R^n \mid W_2(x) \leq \rho\}$. This shows that the origin is globally uniformly stable. □

Theorem 14.2 Let $V:[0,\infty)\times D\to R$ be of C^1 such that

$$W_1(x)\le V(t,x)\le W_2(x)\,; \tag{14.5}$$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\le -W_3(x),\quad \forall\, t\ge 0,\quad \forall\, x\in D, \tag{14.7}$$

where $W_j(x)$ ($j=1,2,3$) are all positive definite. Then, $x=0$ of (14.1) is UAS. If $D=R^n$, and $W_1(x)$ is radially unbounded, then the origin is UGAS.

Proof. We go on with the proof of Theorem 14.1, we know that $x(t;t_0,x_0)\in B_r$ and $x(t;t_0,x_0)$ is defined for all $t\ge t_0$ if $x_0\in\{x\in B_r\,|\,W_2(x)\le\rho\}$. By Lemma 14.4, there exist $\alpha_3\in K$, defined on $[0,r]$, such that $W_3(x)\ge\alpha_3(\|x\|)$.

Hence, we have

$$\alpha_1(\|x\|)\le V(t,x)\le\alpha_2(\|x\|)\,;$$

$$V'(t,x)\le-\alpha_3(\|x\|)\,.$$

Consequently,

$$V' \le -\alpha_3(\|x\|) \le -\alpha_3(\alpha_2^{-1}(V)) = -\alpha(V).$$

Assume, without loss of generality, that $\alpha(\cdot)$ is locally Lip. Let $y(t)$ be the solution of

$$y' = -\alpha(y), \quad y(t_0) = V(t_0, x_0) \ge 0.$$

By the comparison principle, we have

$$V(t, x(t; t_0, x_0)) \le y(t), \quad \forall \ t \ge t_0.$$

By Lemma 14.3, there exists $\sigma(r,s) \in KL$ defined on $[0,a) \times [0,\infty)$ such that

$$V(t, x(t; t_0, x_0)) \le y(t) \le \sigma(V(t_0, x_0)), t - t_0), \text{ for any } V(t_0, x_0) \in [0, \rho].$$

Therefore, any solution starting in $\Omega_{t,\rho}$ satisfies the inequality

$$\|x(t; t_0, x_0)\| \le \alpha_1^{-1}(V(t, x(t; t_0, x_0))) \le \alpha_1^{-1}(\sigma(V(t_0, x_0), t - t_0))$$

$$\le \alpha_1^{-1}(\sigma(\alpha_2(\|x_0\|), t - t_0)) = \beta(\|x_0\|, t - t_0).$$

By Lemma 14.1 it shows that $\beta(\cdot,\cdot)$ is of class KL function. Thus, the inequality (14.3) is satisfied for all $x_0 \in \{x \in B_r \mid W_2(x) \le \rho\}$, which implies that $x = 0$ is UAS.

If $D = R^n$, $W_1(x)$ is radially unbounded, so is $W_2(x)$ by (14.5). Therefore, we can find $\alpha_2 \in K_\infty$ s.t. $W_2(x) \le \alpha_2(\|x\|)$. For any $x_0 \in R^n$, we choose $\rho > 0$ such that $\|x_0\| \le \alpha_2^{-1}(\rho)$. Then, $x_0 \in \{x \in R^n \mid W_2(x) \le \rho\}$. The rest of the proof is the same as the above part for showing UAS. □

Corollary 14.1 Suppose all the assumptions of Theorem 14.2 are satisfied with

$$W_1(x) \geq k_1 \|x\|^c, \; W_2(x) \leq k_2 \|x\|^c, \; W_3(x) \geq k_3 \|x\|^c$$

for $k_j > 0$, and $c > 0$. Then, $x = 0$ is ES. Moreover, if the assumptions hold globally, then $x = 0$ is GES.

Proof. V and V' satisfy the inequalities

$$k_1 \|x\|^c \leq V(t,x) \leq k_2 \|x\|^c;$$

$$V'(t,x) \leq -k_3 \|x\|^c \leq -\frac{k_3}{k_2} V(t,x).$$

By the comparison lemma,

$$V(t,x(t)) \leq V(t_0, x(t_0)) \exp\{-\frac{k_3}{k_2}(t-t_0)\}.$$

Hence,

$$\| x(t) \| \leq \left(\frac{V(t, x(t))}{k_1} \right)^{\frac{1}{c}} \leq \left(\frac{V(t_0, x(t_0)) \exp\{-\frac{k_3}{k_2}(t - t_0)\}}{k_1} \right)^{\frac{1}{c}}$$

$$\leq \left(\frac{k_2 \| x(t_0) \|^c \exp\{-\frac{k_3}{k_2}(t - t_0)\}}{k_1} \right)^{\frac{1}{c}} = \left(\frac{k_2}{k_1}\right)^{\frac{1}{c}} \| x(t_0) \| \exp\{-\frac{k_3}{k_2}(t - t_0)\} .$$

Hence, the origin is ES. If all the assumptions hold globally, the above inequality holds for all $x(t_0) \in R^n$. □

Example 14.3 Consider

$$\dot{x} = (1 + g(t))\,x^3$$

where $g(t)$ is continuous and $g(t) \geq 0$ for all $t \geq 0$. Using the Lyapunov function candidate $V(x) = \frac{1}{2}x^2 > 0$, we obtain

$$V' = -(1 + g(t))\,x^4 \leq x^4 < 0, \quad \forall\, x \in R, \quad \forall\, t \geq 0.$$

Hence, the origin is UGAS.

Example 14.4 Consider

$$\begin{cases} x_1' = -x_1 - g(t)x_2 \\ x_2' = x_1 - x_2 \end{cases}$$

where $g(t)$ is continuously differentiable and satisfies

$$0 \le g(t) \le k \quad \text{and} \quad g'(t) \le g(t), \quad \forall\, t \ge 0.$$

Take $V(t,x) = x_1^2 + (1+g(t))x_2^2$, satisfying

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2.$$

Then, $\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - (2+2g(t)-\dot{g}(t))x_2^2$, Using the inequality

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2, \quad \Rightarrow$$

$$V'(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := -x^T Q x < 0,$$

where Q is positive definite; hence, $V'(t,x)$ is negative definite. The origin is GES.

Converse Lyapunov Theorem

Theorem 14.3 Let $x = 0$ be equilibrium for the time-varying system

$$x' = f(t, x), \tag{14.1}$$

where $f : [0, \infty) \times D_r \to R^n$ is continuous and locally Lip. in x on $D_r = \{x \in R^n \,|\, \|x\| < r\}$.

Assume that for any $(t_0, x_0) \in [0, \infty) \times D_r$, the solution $x(t; t_0, x_0)$ of (14.1) satisfies

$$\| x(t; t_0, x_0) \| \le \beta(\| x_0 \|, t - t_0), \quad \forall\, x_0 \in D_{r_0}, \quad t \ge t_0, \tag{14.8}$$

where $D_{r_0} = \{x \in R^n \,|\, \|x\| < r_0, \ \beta(r_0, 0) < r\} \subseteq D_r$. Then, there exists a continuous function $V(t, x)$ of C^1 that satisfies

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|); \tag{14.9}$$

$$V'(t, x) \le -\alpha_3(\|x\|), \tag{14.10}$$

where $\alpha_j \in K$ ($j = 1, 2, 3$). Moreover, if (14.8) holds globally, then, (14.9) and (14.10) hold globally, where $\alpha_j \in K_\infty$.

Proof. First of all, for any $x_0 \in D_{r_0}$, $x(t; t_0, x_0) \in D_r$ for all $t \geq t_0$. D_{r_0} is a suitable region of attraction.

For any $\beta(s,t) \in KL$, there exists α, $\gamma \in K$, (K_∞ for global case) s.t.

$$\alpha(\beta(s,t)) \leq \gamma(s) \exp\{-t\}. \qquad (14.11)$$

Remark 14.8 (14.11) is a very useful estimate for analysis of Lyapunov stability. It was shown by E. Sontag. Since my proof of this converse theorem is nontrivial, different from the traditional one. My proof is nonlocal constrained.

Define $V(t,x)$ as follows.

$$V(t,x) := \sup_{t_0 \ge 0} \alpha(\| x(t+t_0; t,x) \|) \exp\{\frac{1}{2} t_0\},$$

where $x(t+t_0; t,x)$ is a solution of (14.1). Taking $t_0 = 0$ yields

$$V(t,x) \ge \alpha(\| x(t;t,x) \|) = \alpha(\| x \|) := \alpha_1(\| x \|).$$

Meanwhile, by (14.11), we have

$$V(t,x) \le \sup_{t_0 \ge 0} \alpha(\beta(\| x \|, t_0)) \exp\{\frac{1}{2} t_0\} \le \gamma(\| x \|) \sup_{t_0 \ge 0} \exp\{-\frac{1}{2} t_0\} \le \gamma(\| x \|) := \alpha_2(\| x \|).$$

The derivative of $V(t,x)$ along trajectories of (14.1) is given

$$V'(t,x) = \lim_{h \to 0} \frac{V(t+h, x(t+h;t,x)) - V(t, x(t;t,x))}{h} = \lim_{h \to 0} \frac{V(t+h, \bar{x}) - V(t,x)}{h},$$

where $\bar{x} = x(t+h; t,x)$. Since

$$V(t,\bar{x}) := \sup_{t_0 \geq 0} \alpha(\| x(t+h+t_0; t+h, \bar{x}) \|) \exp\{\frac{1}{2} t_0\}$$

$$= \sup_{t_0 \geq 0} \alpha(\| x(t+h+t_0; t, x) \|) \exp\{\frac{1}{2} t_0\}$$

$$\leq \sup_{t_0 \geq 0} \alpha(\| x(t+t_0; t, x) \|) \exp\{\frac{1}{2} t_0\} \exp\{-\frac{1}{2} h\}$$

$$= V(t,x) \exp\{-\frac{1}{2} h\},$$

we have

$$V'(t,x) = \lim_{h \to 0} \frac{V(t+h,\bar{x}) - V(t,x)}{h} \leq \lim_{h \to 0} V(t,x) \frac{\exp\{-\frac{1}{2} h\} - 1}{h} = -\frac{1}{2} V(t,x)$$

$$\leq -\frac{1}{2} \alpha_1(\| x \|) := -\alpha_3(\| x \|).$$

If (14.8) holds globally, obviously $\alpha_j \in K_\infty$, therefore, (14.9) and (14.10) hold globally. If (14.1) is an autonomous system, we have $x(t_0; t, x) = x(t_0 - t; x)$. Then,

$$V(t, x) := \sup_{t_0 \geq 0} \alpha(\| x(t + t_0; t, x) \|) \exp\{\frac{1}{2} t_0\} = \sup_{t_0 \geq 0} \alpha(\| x(t_0; x) \|) \exp\{\frac{1}{2} t_0\} = V(x),$$

is independent of t. □

Remark 14.9 Compare to the traditional proof, this proof is nonlocal and not so complicated. Although the smooth requirement of f is decreased, the smooth of $V(t,x)$ is also decreased. However, this disadvantage can make up by using the method provided by Yuandan Lin, E.D. Sontag and Yuan Wang for the following paper.

Yuandan Lin, E.D. Sontag and Yuan Wang, "A Smooth Converse Lyapunov Theorem for Robust Stability" SIAM J. Control and Optimization, vol. 34, no.1 pp. 124-160, 1996.

Summary

- Class K and KL functions are useful tools in stability analysis of nonlinear systems.

- Uniform asymptotic stability can be proved using $V(t,x)$ that may dependent on t. However, there should exist α_1, α_2, and $\alpha_3 \in K$, such that

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|);$$

$$V'(t,x) \le -\alpha_3(\|x\|).$$

So the bounding functions α_j are independent of t.

Homework

1. Show Lemma 14.3.
2. Show Lemma 14.4.